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Weighted L_p Error of Lagrange Interpolation*

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The authors give new error estimates of Lagrange interpolation in the weighted $L_{p,u}$ norm, when u is a generalized Jacobi weight and the interpolation points are the zeros of polynomials orthogonal with respect to (another) generalized Jacobi weight. © 1995 Academic Press, Inc.

1. INTRODUCTION

Let $X = \{x_{m,k},\ k = 1, ..., m,\ m = 1, 2, ...\} \subset (-1, 1)$ be a matrix of knots and let f be bounded function on $[-1, 1]$. We denote by $L_m(X, f)$ the Lagrange polynomial interpolating the function f at $x_{m,k}$, $k = 1, ..., m$. The operator $L_m(X)$ maps bounded functions into continuous functions with an L_p weighted norm, $1 \leqslant p < \infty$. Therefore, if u is a suitable weight function, then

$$\|L_m(X, f)\, u\|_p \leqslant \text{const } \|f\|_\infty$$

holds. Indeed, when the entries of X are the zeros of certain orthogonal polynomials, then there are presented in the literature necessary and sufficient conditions on u for the above to hold (see e.g. [10, 14]. There also are some necessary conditions on u when X is a general matrix of

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points dense in $[-1, 1]$ (see [8]). In particular, if f is a continuous function, using the previous bound we get

$$\|[f - L_m(X, f)]\, u\|_p \leqslant \text{const}\ E_{m-1}(f),$$

where $E_m(f)$ denotes the error of the best uniform approximation by algebraic polynomials. Nevertheless, the last estimate often is not suitable (if, say, f is not continuous). Further, as it turns out, in many applications it is necessary to estimate the interpolation error in an L_p weighted norm by the same norm of the (local) derivative of the function f. In the present paper, we obtain estimates of this kind, when $p \in (1, \infty)$, u is a generalized Jacobi weight, and the points of interpolation are the zeros of the generalized Jacobi polynomials. First, in the last inequality, we replace $E_{m-1}(f)$ by the error $\tilde{E}_{m-1}(f)_{u,p}$ of the best one-sided approximation in the L_p space with weight u. Subsequently, we give estimates of $\tilde{E}_{m-1}(f)_{u,p}$ when the function f is locally absolutely continuous. This procedure can be applied to several discrete type operators.

2. Main Results

We say that $f \in L_p([a, b])$, $-1 \leqslant a < b \leqslant 1$, $1 \leqslant p < \infty$, if and only if

$$\|f\|^p_{L_p([a,b])} = \int_a^b |f(x)|^p\, dx < \infty.$$

If $a = -1$ and $b = 1$, then we write $f \in L_p$ and $\|f\|^p_p = \int_{-1}^1 |f(x)|^p\, dx$. If $p = \infty$, we consider the *vraisup* norm. Further, we denote by AC_{Loc} the class of the functions absolutely continuous in any closed set $[a, b] \subset (-1, 1)$. In the following Π_m denotes the set of the polynomials of degree at most m. Throughout this paper, the symbol "$\mathscr{C}$" stands for a positive constant which may take different values on different occurrences. Let g be a bounded and measurable function, and let σ be a weight function with $\sigma \in L_p$. We set

$$\begin{aligned} \tilde{E}_m(g)_{\sigma,p} = \inf\{\|(Q^+ - Q^-)\,\sigma\|_p, \qquad & Q^\pm \in \Pi_m, \\ Q^-(x) \leqslant g(x) \leqslant Q^+(x), \qquad & x \in [-1, 1]\}. \end{aligned} \tag{2.1}$$

$\tilde{E}_m(g)_{\sigma,p}$ is called the error of the best one-sided approximation of the function g in L_p space with weight σ.

Let

$$w(x) = \psi(x)\, v^{\alpha,\beta}(x) \prod_{k=1}^{s} |\chi_k - x|^{\gamma_k}, \qquad |x| \leqslant 1, \tag{2.2}$$

where $v^{\alpha,\beta}(x)=(1-x)^{\alpha}(1+x)^{\beta}$, $\alpha, \beta>-1$ is a Jacobi weight, $\gamma_k>-1$, $|\chi_k|<1$, $k=1, ..., s$, $0<\psi\in \mathrm{Lip}\,\lambda$, $0<\lambda\leqslant 1$. The weight w is called generalized smooth Jacobi weight ($w\in \mathrm{GSJ}$) (see e.g. [1, 10]). Now, let $\{p_m(w))\}_{m\in\mathbb{N}}$ be the system of orthonormal polynomials corresponding to the weight function w, that is, $p_m(w)$ is a polynomial of degree exactly m with positive leading coefficient and $\int_{-1}^{1} p_m(w; t)\, p_n(w; t)\, w(t)\, dt=\delta_{m,n}$. We denote by $x_k=x_{m,k}(w)$, $k=1, ..., m$, the zeros of $p_m(w)$ indexed in increasing order and by $L_m(w, f)$ the Lagrange polynomial interpolating a given function f: $(-1, 1)\to\mathbb{R}$ at x_k, $k=1, ..., m$. (Incidentally, the function f can be unbounded at ± 1.) Thus, setting

$$f_m(x)=\begin{cases} f(x_1) & \text{if } x\in(-\infty, x_1], \\ f(x) & \text{if } x\in[x_1, x_m], \quad m\geqslant 1, \\ f(x_m) & \text{if } x\in[x_m, \infty), \end{cases}$$

we have $L_m(w, f_m)=L_m(w, f)$ and

$$\|[f-L_m(w, f)]\,u\|_p\leqslant\|(f-f_m)\,u\|_p+\|[f_m-L_m(w, f_m)]\,u\|_p. \tag{2.3}$$

In most cases u is a generalized Jacobi weight ($u\in \mathrm{GJ}$), i.e.,

$$u(x)=v^{\gamma,\delta}(x)\prod_{k=1}^{r}|\tau_k-x|^{a_k}, \qquad |x|\leqslant 1. \tag{2.4}$$

In the following we use notations $\varphi(x)=\sqrt{1-x^2}$ and $u_m(x)=v^{\gamma,\delta}(x)\prod_{k=1}^{r}(|\tau_k-x|+m^{-1})^{a_k}$.

THEOREM 2.1. *Let $p\in(1, \infty)$, $q=p/(p-1)$ and $m=2, 3, ...$. If the weight functions $u\in \mathrm{GJ}$, $w\in \mathrm{GSJ}$ satisfy the conditions*

$$u\in L_p, \quad \frac{u}{\sqrt{w}}\varphi^{-1/2}\in L_p, \quad \textit{and} \quad \frac{\sqrt{w}}{u}\varphi^{-1/2}\in L_q, \quad \frac{w}{u}\in L_q, \tag{2.5}$$

then for every bounded and measurable function g: $[-1, 1]\to\mathbb{R}$

$$\|[g-L_m(w, g)]\,u\|_p\leqslant\mathscr{C}\tilde{E}_{m-1}(g)_{u,p}, \tag{2.6}$$

where $\mathscr{C}$ is a positive constant independent of m and g.

The next crucial step is to estimate $\tilde{E}_m(f_m)_{u,p}$ (f_m is the function defined above).

THEOREM 2.2. *Let* $p \in [1, \infty]$, $m = 2, 3, \ldots$, *and* $u \in \mathrm{GJ}$, $w \in \mathrm{GSJ}$. *If* $u \in L_p$ *and the function* $f \in \mathrm{Ac}_{\mathrm{Loc}}$ *satisfies*

$$f' \varphi u_m \in L_p([x_1, x_m]), \tag{2.7}$$

where $x_1 = x_{m,1}(w)$ *and* $x_m = x_{m,m}(w)$, *then*

$$\tilde{E}_m(f_m)_{u,p} \leq \frac{\mathscr{C}}{m} \|\varphi f' u_m\|_{L_p([x_1, x_m])}, \tag{2.8}$$

where $\mathscr{C}$ *is a positive constant independent of* m, p, *and* f.

Theorem 2.2 yields

COROLLARY 2.3. *Using the assumptions and the notations of the Theorem 2.2, we get*

$$\tilde{E}_m(f_m)_{u,p} \leq \frac{\mathscr{C}}{m} \tilde{E}_{m-1}(f'_m)_{u\varphi,p}, \qquad m \geq 2, \tag{2.9}$$

where $\mathscr{C}$ *is a positive constant independent of* m, p, *and* f.

The iterated application of (2.9) and (2.8) gives that if $f^{(r)}\varphi^r u_m \in L_p([x_1, x_m])$, then

$$\tilde{E}_m(f_m)_{u,p} \leq \frac{\mathscr{C}}{m^r} \|f^{(r)}\varphi^r u_m\|_{L_p([x_1, x_m])}, \qquad r \geq 1, \quad m \geq 2. \tag{2.10}$$

When $u(x) \equiv 1$ and $1 \leq p \leq \infty$, estimates similar to (2.10) are in [15] and [5]. Further, when $p = 1$ and $u = \sigma(1 + \log^+ \sigma)$, $\sigma \in \mathrm{GJ}$, estimates of $\tilde{E}_m(f)_{u,1}$ can be found in [7]. Now we estimate the error of Lagrange interpolation. Setting $I_m = [-1, x_1] \cup [x_m, 1]$, the following theorem holds.

THEOREM 2.4. *Let* $p \in (1, \infty)$ *and* $m = 2, 3, \ldots$. *Assume that* $u \in \mathrm{GJ}$ *and* $w \in \mathrm{GSJ}$ *satisfy* (2.5). *If* $u \in L_p$, *then for every function* $f \in \mathrm{AC}_{\mathrm{Loc}}$ *with* $f' \varphi^{2/p} u \in L_1$

$$\|[f - L_m(w, f)]\, u\|_p \leq \mathscr{C} \left[\tilde{E}_{m-1}(f_m)_{u,p} + \int_{I_m} |f'(t)| (1 - t^2)^{1/p}\, u(t)\, dt \right], \tag{2.11}$$

where $\mathscr{C}$ *is a positive constant independent of* m *and* f.

Remark 1. Using a similar argument, we can prove inequality (2.11) if we replace u by u_m.

The estimate (2.11) is useful when the function f is unbounded at ± 1. For instance, let $w(x) = (1-x^2)^\alpha$ and $u(x) = (1-x^2)^\gamma$. In this case (2.5) means

$$\gamma > -\frac{1}{p} \quad \text{and} \quad \max\left(\gamma - 1 + \frac{1}{p}, 2\gamma - \frac{3}{2} + \frac{2}{p}\right) < \alpha < 2\gamma - \frac{1}{2} + \frac{2}{p}.$$

Then, by (2.10) and (2.11) we obtain

$$\|[f - L_m(w, f)]\, u\|_p = \begin{cases} \mathcal{O}(m^{-2\sigma - 2\gamma - 2/p}) \\ \quad \text{if } f(x) = (1-x)^\sigma, \sigma + \gamma + 1/p > 0, \\ \mathcal{O}(m^{-2\gamma - 2/p}) \\ \quad \text{if } f(x) = \log(1+x). \end{cases}$$

The previous estimates have the same order as the best approximation in L_p space with Jacobi weight (see [4, pp. 109, 110]).

Now we can state the following.

COROLLARY 2.5. *Let* $p \in (1, \infty)$, $m = 2, 3, ...,$ *assume that* $u \in \mathrm{GJ}$, $w \in \mathrm{GSJ}$ *satisfy* (2.5). *If* $u \in L_p$, $f \in \mathrm{AC}_{\mathrm{Loc}}$ *and* $f' \varphi u_m \in L_p$, *then*

$$\begin{aligned} \|[f - L_m(w, f)]\, u\|_p &\leqslant \frac{\mathscr{C}}{m} \|f' \varphi u_m\|_p, \\ \|[f - L_m(w, f)]\, u_m\|_p &\leqslant \frac{\mathscr{C}}{m} \|f' \varphi u_m\|_p, \end{aligned} \tag{2.12}$$

where $\mathscr{C}$ *is a positive constant independent of* m *and* f.

Inequality (2.11), together with (2.10) and (2.12), is sufficient to estimate the weighted L_p interpolation error for a wide class of functions. In fact, from (2.12) it follows, whenever $f' \varphi u_m \in L_p$, that

$$\|[f - L_m(w, f)]\, u\|_p \leqslant \frac{\mathscr{C}}{m} E_{m-2}(f')_{\varphi u_m, p}, \tag{2.13}$$

where $E_m(g)_{\sigma, p} = \inf_{P \in \Pi_m} \|(g - P)\, \sigma\|_p$ and σ is a weight function. (Indeed, we apply (2.12) for the function $F(x) = f(x) - \int_{-1}^{x} P_{m-2}(t)\, dt$, where $P_{m-2} \in \Pi_{m-2}$.)

If $u = v^{\gamma, \delta}$ (i.e., $a_k = 0$, $k = 1, ..., r$), we can estimate $E_{m-2}(f')_{\varphi u_m, p}$ by the main part of the φ-modulus of continuity [3]. Unfortunately, (2.13) does not work in the general case $u \neq v^{\gamma, \delta}$ because, at present, estimates of $E_{m-2}(f')_{\varphi u_m, p}$ are not available. Instead we can proceed as follows. Replacing in (2.12) f by $f(x) - \int_{-1}^{x} L_{m-1}(w\varphi^2, f', t)\, dt$, we have

$$\|[f - L_m(w, f)]\, u\|_p \leqslant \frac{\mathscr{C}}{m} \|[f' - L_{m-1}(w\varphi^2, f')]\, \varphi u_m\|_p, \tag{2.14}$$

$$\|[f - L_m(w, f)]\, u_m\|_p \leqslant \frac{\mathscr{C}}{m} \|[f' - L_{m-1}(w\varphi^2, f')]\, \varphi u_m\|_p, \tag{2.15}$$

with $1 < p < \infty$.

Moreover, if $u \in \mathrm{GJ}$ and $w \in \mathrm{GSJ}$ satisfy (2.5), then so do $u\varphi^k$ and $w\varphi^{2k}$. Therefore, starting from (2.14) we can iterate (2.15) and finally apply the second inequality in (2.12) or (2.11) together with (2.10). Hence, for instance, we get

$$\|[f - L_m(w, f)]\, u\|_p \leqslant \frac{\mathscr{C}}{m^r} \|\varphi^r f^{(r)} u_m\|_p, \qquad r \geqslant 1, \quad 1 < p < \infty. \tag{2.16}$$

In particular, if $p = 2$, $w \in \mathrm{GSJ}$ and $u = \sqrt{w}$, the conditions (2.5) are satisfied. So the previous estimates are refinements of the well-known theorem of Erdős and Turán [4] for GSJ weights.

Finally, we get

COROLLARY 2.6. *Let* $p \in [1, \infty]$, $m = 2, 3, \ldots$, *and assume that* $u \in \mathrm{GJ}$ *and* $u \in L_p$. *If* $f \in \mathrm{AC}_{\mathrm{Loc}}$ *and* $f'\varphi u_m \in L_p$, *then*

$$E_m(f)_{u,p} \leqslant \frac{\mathscr{C}}{m} \|f'\varphi u_m\|_p,$$

where $\mathscr{C}$ *is a positive constant independent of* m, p, *and* f.

Remark 2. It is easy to prove that $E_m(f)_{u_m, p} \leqslant (\mathscr{C}/m)\|f'\varphi u_m\|_p$ (see the proofs in Section 3). Unfortunately, at this moment we cannot prove the estimate $E_m(f)_{u,p} \leqslant (\mathscr{C}/m)\|f'\varphi u\|_p$ with $u \in \mathrm{GJ}$, which holds when $u = v^{\gamma, \delta}$ or when the exponents a_k of the weight u are negative. The case when every $a_k > 0$ is open. Similar remarks hold for the other estimates.

3. PROOFS

Proof of Theorem 2.1. From Theorem 9.25 by Nevai [11, p. 169], we get

Statement A. Let $w \in \mathrm{GSJ}$, $u \in \mathrm{GJ}$, $u \in L_p$, $1 < p < \infty$, and $P \in \Pi_{m-1}$. Then

$$\sum_{k=1}^{m} \lambda_{m,k}(w)\, |u_m(x_{m,k}(w))\, P(x_{m,k}(w))|^p / w_m(x_{m,k}(w)) \leqslant \mathscr{C}\, \|Pu\|_p^p,$$

where

$$w_m(x) = v^{\alpha,\beta}(x) \prod_{k=1}^{s} (|\chi_k - x| + m^{-1})^{\gamma_k},$$

$$u_m(x) = v^{\gamma,\delta}(x) \prod_{k=1}^{r} (|\tau_k - x| + m^{-1})^{a_k}, \qquad |x| < 1,$$

$$\lambda_{m,k}(w) = \left[\sum_{i=0}^{m-1} p_i^2(w, x_{m,k}(w)) \right]^{-1},$$

and $\mathscr{C} = \mathscr{C}(w, u, p)$.

We also need a consequence of Theorem 3.2 by Xu [17, p. 82].

Statement B. By the notations and conditions of Theorem 2.1 and Statement A, we have, for $P \in \Pi_{m-1}$,

$$\|Pu\|_p^p \leqslant \mathscr{C} \sum_{k=1}^{m} \lambda_{m,k}(w)\, |u_m(x_{m,k}(w))\, P(x_{m,k}(w))|^p / w_m(x_{m,k}(w)).$$

In Statement B we applied the cast (replacing u in Xu's theorem by V) $V = u^p/w$, $\beta' = u$, and $\alpha' = w$.

Let g be a bounded and measurable function and $Q^{\pm} \in \Pi_{m-1}$ such that $Q^-(x) \leqslant g(x) \leqslant Q^+(x)$, $|x| \leqslant 1$.

With $u \in \mathrm{GJ}$, we have

$$\|[g - L_m(w, g)]\, u\|_p \leqslant \|[Q^+ - Q^-]\, u\|_p + \|L_m(w, f - Q^-)\, u\|_p.$$

Using Statement B and then Statement A,

$$\begin{aligned} \|L_m(w, f - Q^-)\, u\|_p^p &\leqslant \mathscr{C} \sum_{k=1}^{m} \lambda_{m,k}(w) \frac{u_m^p(x_{m,k}(w))}{w_m(x_{m,k}(w))} [f - Q^-]^p (x_{m,k}(w)) \\ &\leqslant \mathscr{C} \sum_{k=1}^{m} \lambda_{m,k}(w) \frac{u_m^p(x_{m,k}(w))}{w_m(x_{m,k}(w))} [Q^+ - Q^-]^p (x_{m,k}(w)) \\ &\leqslant \mathscr{C} \|(Q^+ - Q^-)\, u\|_p^p. \end{aligned}$$

Then

$$\|[g - L_m(w, g)]\, u\|_p \leqslant \mathscr{C} \|(Q^+ - Q^-)\, u\|_p, \tag{3.1}$$

and Theorem 2.1 follows from (3.1), making the infimum with respect to $Q^{\pm}$. ∎

In order to prove Theorem 2.2, we need some preliminary facts and lemmas. If A and B are two expressions depending on some variables then we write $A \sim B$ if $|A/B|^{\pm 1} \leqslant \mathscr{C}$ uniformly for the variables under consideration.

LEMMA 3.1. *Let* $u(x) = b(x)\, v^{\gamma,\delta}(x) \prod_{k=1}^{r} |\tau_k - x|^{a_k} \in \mathrm{GJ}$ *and* $y_k = y_{m,k} = -\cos(k\pi/(m+1))$, $k = 0, \ldots, m+1$. *If* $u \in L_p$, *then for every* k *with* $1 \leqslant |k| \leqslant m$ *and* $1 \leqslant v$, $v \pm k$, $v \pm k \pm 1 \leqslant m$,

$$\int_{y_{v-1}}^{y_v} u(x)\, dx \leqslant \mathscr{C} |k|^{\Gamma} \int_{y_{v+k-1}}^{y_{v+k}} u(x)\, dx$$

where $\Gamma = \max\{(|2\gamma+1|, |2\delta+1|\} + 2r \max_{k=1,\ldots,r} |a_k|$ *and* $\mathscr{C}$ *is an absolute constant independent of* m, k, *and* v.

Proof The proof requires examination of several particular cases, but only simple calculations. For the sake of brevity we consider the weight $u(x) = v^{\gamma,\delta}(x)|x - \tau_1|^{a_1}$ and the case

$$-1 \leqslant y_{v-1} \leqslant y_v < \cdots < y_{v+k-1} < y_{v+k} < \cdots < y_{v+k+s} \leqslant \tau_1 \leqslant y_{v+k+s+1} \leqslant 0.$$

Then

$$I_v = \int_{y_{v-1}}^{y_v} u(x)\, dx \sim \frac{v}{m^2} \left(\frac{v}{m}\right)^{2\delta} \left(\frac{(v+k+s)^2 - v^2}{m^2}\right)^{a_1},$$

$$I_{v+k} = \int_{y_{v+k-1}}^{y_{v+k}} u(x)\, dx \sim \frac{v+k}{m^2} \left(\frac{v+k}{m}\right)^{2\delta} \left(\frac{(v+k+s)^2 - (v+k)^2}{m^2}\right)^{a_1}$$

and

$$\frac{I_v}{I_{v+k}} \sim \left(\frac{v}{v+k}\right)^{2\delta+1} \left(\frac{(v+k+s)^2 - v^2}{(v+k+s)^2 - (v+k)^2}\right)^{a_1}$$
$$= \left(\frac{v}{v+k}\right)^{2\delta+1} \left(\frac{k+s}{s}\right)^{a_1} \left(\frac{2v+s+k}{2v+s+2k}\right)^{a_1}.$$

Assume $k > 0$. (If $k < 0$, we can consider I_{v+k}/I_v.) Then, we obtain

$$\left(\frac{v}{v+k}\right)^{2\delta+1} \leqslant \mathscr{C} k^{|2\delta+1|}.$$

Moreover, if $a_1 > 0$, we have

$$\left(\frac{2v+s+k}{2v+s+2k}\right)^{a_1} < 1$$

and

$$\left(\frac{k+s}{s}\right)^{a_1} \leqslant \mathscr{C} k^{a_1}.$$

If $a_1 < 0$, then $((k+s)/s)^{a_1} < 1$ and $((2\nu+s+k)/(2\nu+s+2k))^{a_1} = (1+k/(2\nu+k+s))^{-a_1} < \mathscr{C} k^{-a_1}$. Hence

$$\frac{I_\nu}{I_{\nu+k}} \leqslant \mathscr{C} k^{|2\delta+1|+|a_1|},$$

as was stated. The other cases are similar. ∎

Let $x_k = x_{m,k}(w)$, $k = 1, ..., m$, with $m \geqslant 2$. Putting $n = \mu m$, $2\pi < \mu \in \mathbb{N}$, we denote by $t_k = t_{n+1,k} = -\cos(((2k-1)/(n+1))(\pi/2))$, $k = 1, ..., n+1$, the zeros of the Chebyshev polynomial T_{n+1}. Since $w \in \mathrm{GSJ}$, $1 + x_{m,1}(w) \sim m^{-2} \sim 1 - x_{m,m}(w)$; hence there exists a fixed $\bar{\mu} \in \mathbb{N}$ such that, for $\mu > \max(2\pi, \bar{\mu})$ we have

$$-1 < t_1 < \cdots < t_{\rho-1} \leqslant x_1 < t_\rho < \cdots < t_\sigma < x_m \leqslant t_{\sigma+1} < \cdots < t_{n+1} < 1,$$

for some $\rho > 1$ and $\sigma \leqslant n$.

Now we define the function $S^+(x) = S^+(f_m, x)$ as

$$S^+(f_m, x) = M_\rho + \sum_{k=\rho}^{\sigma} (x - t_k)_+^0 \, \delta_k,$$

where generally

$$\eta_t(x) = (x-t)_+^0 = \begin{cases} 0, & x \leqslant t, \\ 1, & x > t. \end{cases}$$

Furthermore,

$$\begin{aligned} M_\rho &= \sup\{f_m(t),\ -1 \leqslant t \leqslant t_\rho\}, \\ M_k &= \sup\{f_m(t),\ t_{k-1} < t \leqslant t_k\}, \qquad \rho+1 \leqslant k \leqslant \sigma, \\ M_{\sigma+1} &= \sup\{f_m(t),\ t_\sigma < t \leqslant 1\}, \\ \delta_k &= M_{k+1} - M_k, \qquad k = \rho, ..., \sigma. \end{aligned}$$

Analogously, we can define $S^-(x) = S^-(f_m, x)$ as

$$S^-(f_m, x) = m_\rho + \sum_{k=\rho}^{\sigma} (x - t_k)_+^0 \, \bar{\delta}_k,$$

where

$$m_\rho = \inf\{f_m(t),\ -1 \leqslant t \leqslant t_\rho\},$$
$$m_k = \inf\{f_m(t),\ t_{k-1} < t \leqslant t_k\}, \qquad \rho+1 \leqslant k \leqslant \sigma,$$
$$m_{\sigma+1} = \inf\{f_m(t),\ t_\sigma < t \leqslant 1\},$$
$$\bar{\delta}_k = m_{k+1} - m_k, \qquad k = \rho, ..., \sigma.$$

By definition,

$$S^-(f_m, x) \leqslant f_m(x) \leqslant S^+(f_m, x).$$

Now we put $M = 2an$, with $a \in \mathbb{N}$, $2a > \Gamma + 2$, and Γ as defined in Lemma 3.1, and we define the polynomials $P^+_{M,k}, P^-_{m,k} \in \Pi_M$, $\rho \leqslant k \leqslant \sigma$, as follows:

$$P^+_{M,k}(t_i) = \begin{cases} 1, & i = k, k+1, ..., n+1, \\ 0, & i = 1, 2, ..., k-1, \end{cases}$$
$$\frac{d^j}{dx^j} P^+_{M,k}(t_i) = 0, \qquad i \neq k, \quad j = 1, 2, ..., 2a-1,$$
$$P^-_{M,k}(t_i) = \begin{cases} 1, & i = k+1, ..., n+1, \\ 0, & i = 1, 2, ..., k, \end{cases}$$
$$\frac{d^j}{dx^j} P^-_{M,k}(t_i) = 0, \qquad i \neq k, \quad j = 1, 2, ..., 2a-1.$$

Working as in [15], we can prove that

$$P^-_{M,k}(x) \leqslant (x - t_k)^0_+ \leqslant P^+_{M,k}(x)$$

and

$$P^+_{M,k}(x) - P^-_{M,k}(x) = l_k^{2a}(x),$$

where l_k is the kth fundamental Lagrange polynomial based on the Chebyshev zeros $t_1, ..., t_{n+1}$. Moreover, by the previous polynomials, we define

$$P^+_M(x) = \sum_{\delta_k > 0} P^+_{M,k}(x)\,\delta_k + \sum_{\delta_k < 0} P^-_{M,k}(x)\,\delta_k + M_\rho,$$
$$P^-_M(x) = \sum_{\delta_k > 0} P^-_{M,k}(x)\,\delta_k + \sum_{\delta_k < 0} P^+_{M,k}(x)\,\delta_k + M_\rho,$$
$$q^+_M(x) = \sum_{\bar{\delta}_k > 0} P^+_{M,k}(x)\,\bar{\delta}_k + \sum_{\bar{\delta}_k < 0} P^-_{M,k}(x)\,\bar{\delta}_k + m_\rho,$$
$$q^-_M(x) = \sum_{\bar{\delta}_k > 0} P^-_{M,k}(x)\,\bar{\delta}_k + \sum_{\bar{\delta}_k < 0} P^+_{M,k}(x)\,\bar{\delta}_k + m_\rho,$$

for $k=\rho, \ldots, \sigma$, and from the definitions of $S^{\pm}(f_m, x)$ it follows that

$$q_M^-(x) \leqslant S^-(f_m, x) \leqslant q_M^+(x), \qquad P_M^-(x) \leqslant S^+(f_m, x) \leqslant P_M^+(x), \tag{3.2}$$

$$q_M^+(x) - q_M^-(x) = \sum_{k=\rho}^{\sigma} l_k^{2a}(x) |\bar{\delta}_k|, \tag{3.3}$$

$$P_M^+(x) - P_M^-(x) = \sum_{k=\rho}^{\sigma} l_k^{2a}(x) |\delta_k|. \tag{3.4}$$

We still need a definition. Letting $y_k = y_{n,k} = -\cos(k\pi/(n+1))$, $k = 1, 2, \ldots, n$, be the zeros of the nth Chebyshev polynomial of the second kind, U_n, it is well-known that $t_k < y_k < t_{k+1}$, $k=1, \ldots, n$. Let us define $\bar{S}^{\pm}(f_m)$ by

$$\bar{S}^+(f_m, x) = \begin{cases} |\delta_k|, & y_{k-1} \leqslant x \leqslant y_k, \quad k = \rho, \ldots, \sigma, \\ 0, & x < y_{\rho-1} \quad \text{or} \quad y_\sigma < x, \end{cases}$$

$$\bar{S}^-(f_m, x) = \begin{cases} |\bar{\delta}_k|, & y_{k-1} \leqslant x \leqslant y_k, \quad k = \rho, \ldots, \sigma, \\ 0, & x < y_{\rho-1} \quad \text{or} \quad y_\sigma < x. \end{cases}$$

Now we prove the following.

LEMMA 3.2. *Let $u \in \mathrm{GJ}$ be defined by (2.4) and $u \in L_p$ with $1 \leqslant p \leqslant \infty$. If f_m is bounded and measurable, then*

$$\|(P_M^+ - P_M^-)\, u\|_p \leqslant \mathscr{C}\, \|\bar{S}^+(f_m)\, u\|_{L_p([y_{\rho-1}, y_\sigma])}, \tag{3.5}$$

$$\|(q_M^+ - q_M^-)\, u\|_p \leqslant \mathscr{C}\, \|\bar{S}^-(f_m)\, u\|_{L_p([y_{\rho-1}, y_\sigma])}, \tag{3.6}$$

where $\mathscr{C}$ is a positive constant independent of p, f and $m > 5$.

Proof. For the sake of brevity, we prove (3.5). Formula (3.6) can be proved similarly. We observe that if $x \in [y_{i-1}, y_i]$, $i = \rho, \ldots, \sigma$, then

$$l_k^{2a}(x) \leqslant \frac{\mathscr{C}}{(|k+i|+1)^{2a}}.$$

Hence, from (3.4) and the Hölder inequality

$$|P_M^+(x) - P_M^-(x)|^p\, u^p(x) \leqslant \mathscr{C}^p \left(\sum_{k=\rho}^{\sigma} \frac{|\delta_k|\, u(x)}{(|k-i|+1)^{2a}} \right)^p$$

$$\leqslant \mathscr{C}^p \sum_{k=\rho}^{\sigma} \frac{|\delta_k|^p\, u^p(x)}{(|k-i|+1)^{(2a-1)p}}.$$

On the other hand, recalling Lemma 3.1 and the definition of $\bar{S}^+(f_m)$, we have

$$\int_{y_{i-1}}^{y_i} |\delta_k|^p u^p(x)\,dx \leqslant \mathscr{C}(|k-i|+1)^{p\Gamma} \int_{y_{k-1}}^{y_k} |\delta_k|^p u^p(x)\,dx$$
$$= \mathscr{C}(|k-i|+1)^{p\Gamma} \int_{y_{k-1}}^{y_k} \bar{S}^+(f_m, x)^p u^p(x)\,dx.$$

Then it follows that

$$\int_{y_{i-1}}^{y_i} [P_M^+(x) - P_M^-(x)]^p u^p(x)\,dx \leqslant C^p \sum_{k=\rho}^{\sigma} (|k-i|+1)^{-(2a-1-\Gamma)p}$$
$$\times \int_{y_{k-1}}^{y_k} \bar{S}^+(f_m, x)^p u^p(x)\,dx.$$

Finally, by summing on $i = 1, ..., m+1$, $y_0 = -1$, $y_{m+1} = 1$, we have

$$\|(P_M^+ - P_M^-)\,u\|_p^p \leqslant \mathscr{C}^p \sum_{k=\rho}^{\sigma} \int_{y_{k-1}}^{y_k} \bar{S}^+(f_m, x)^p u^p(x)\,dx$$
$$\times \left\{ \sum_{i=1}^{m+1} (|k-i|+1)^{-(2a-1-\Gamma)p} \right\},$$

whence by $2a - \Gamma - 1 > 1$ the above sum $\{\cdots\}$ is bounded for any k, so

$$\|(P_M^+ - P_M^-)\,u\|_p \leqslant \mathscr{C}\, \|\bar{S}^+(f_m)\,u\|_{L^p([y_{\rho-1}, y_\sigma])}. \quad \blacksquare$$

The following lemmas estimate the functions $\bar{S}^\pm(f_m)$.

LEMMA 3.3. *If f is locally absolutely continuous and $x \in [-1, 1]$, then*

$$\bar{S}^\pm(f_m, x) \leqslant \int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)} |f'_m(t)|\,dt,$$

where $\Delta_m(x) = (\sqrt{1-x^2}/m) + (1/m^2)$.

Proof. We prove the lemma for $\bar{S}^+(f_m)$, say. If $x \notin [y_{\rho-1}, y_\sigma]$, $S^+(f_m, x) = 0$, so the statement is trivial. Now we assume $x \in [y_{\rho-1}, y_\rho]$. By $t_{\rho-1} < y_{\rho-1} < t_\rho < y_\rho < t_{\rho+1}$,

$$\begin{aligned}\bar{S}^+(f_m, x) &= |\delta_\rho| = |M_{\rho+1} - M_\rho| \\ &\leqslant \sup\{|f_m(t) - f_m(t')|, \ -1 \leqslant t, t' \leqslant t_{\rho+1}\} \\ &\leqslant \int_{-1}^{t_{\rho+1}} |f'_m(t)|\, dt = \int_{\tau_{\rho-1}}^{t_{\rho+1}} |f'_m(t)|\, dt,\end{aligned}$$

since $f'_m(x) = 0$ if $x < x_1$ or $x > x_m$. Now, for every $x \in [y_{k-1}, y_k]$, $k = \rho, ..., \sigma$, it results that

$$t_{k+1} - t_{k-1} \leqslant \frac{\pi}{n}\sqrt{1-x^2} + \frac{1}{2}\left(\frac{\pi}{n}\right)^2.$$

Being $n = \mu m$ with $\mu > 2\pi$, we have

$$t_{k+1} - t_{k-1} \leqslant \frac{\sqrt{1-x^2}}{2m} + \frac{1}{2m^2} = \frac{\Delta_m(x)}{2}.$$

Therefore

$$\bar{S}^+(f_m, x) \leqslant \int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)} |f'_m(t)|\, dt, \qquad x \in [y_{\rho-1}, y_\rho].$$

Similar argument works for $x \in [y_{\sigma-1}, y_\sigma]$. If $x \in [y_{k-1}, y_k]$, $k = \rho + 1, ..., \sigma - 1$, then

$$\begin{aligned}\bar{S}^+(f_m, x) &= |\delta_k| = |M_{k+1} - M_k| \\ &\leqslant \sup\{|f_m(t) - f_m(t')|, \qquad t, t' \in [t_{k-1} t_{k+1}]\} \\ &\leqslant \int_{t_{k-1}}^{t_{k+1}} |f'_m(t)|\, dt \leqslant \int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)} |f'_m(t)|\, dt,\end{aligned}$$

and the lemma is proved. ∎

Now, as before letting $u_m(x) = v^{\gamma, \delta}(x) \prod_{k=1}^{r} (|x - \tau_k| + m^{-1})^{a_k}$, $\varphi(x) = \sqrt{1-x^2}$, we prove the following.

LEMMA 3.4. *Let $u \in \mathrm{GJ}$ be defined by (2.4) and $u \in L_p$ with $1 \leqslant p \leqslant \infty$. If $f \in \mathrm{AC_{Loc}}$ and $\varphi f' u_m \in L_p([x_1, x_m])$, then, for $m \geqslant 2$,*

$$\|\bar{S}^{\pm}(f_m)\, u\|_{L_p([y_{\rho-1}, y_\sigma])} \leqslant \frac{\mathscr{C}}{m} \|\varphi f' u_m\|_{L_p([x_1, x_m])},$$

where $\mathscr{C}$ is a constant independent of p, f, and m.

Proof. For the sake of simplicity we prove the theorem for $\bar{S}^+(f_m)$ and with $u(x)=v^{\gamma,\delta}(x)|x-\tau_1|^{a_1}$. Now, by Lemma 3.3 and $f'_m(x)=0$, $x\notin[x_1, x_m]$, for $p<\infty$, we have

$$\|\bar{S}^+(f_m)\,u\|^p_{L_p([y_{\rho-1}, y_\sigma])}=\int_{y_{\rho-1}}^{x_1}\cdots+\int_{x_1}^{x_m}\cdots+\int_{x_m}^{y_\sigma}\cdots$$

$$\leqslant\int_{x_1}^{x_m}\left[\int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)}|f'_m(t)|\,dt\right]^p u^p(x)\,dx$$

$$=\int_{A_m}\left[\int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)}|f'_m(t)|\,dt\right]^p u^p(x)\,dx$$

$$+\int_{\tau_1-m^{-1}}^{\tau_1+m^{-1}}\left[\int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)}|f'_m(t)|\,dt\right]^p u^p(x)\,dx$$

$$=I_1+I_2,$$

with $A_m=[x_1, \tau_1-m^{-1}]\cup[\tau_1+m^{-1}, x_m]$.

To estimate I_1, we observe that, x being an element of A_m and $|x-t|\leqslant \Delta_m(x)$ by [3, p. 80], $u(x)\sim u_m(x)\leqslant\mathscr{C}u_m(t)$, $\Delta_m(x)\leqslant\mathscr{C}\Delta_m(t)$, and $\Delta_m(t)\leqslant \mathscr{C}(\sqrt{1-t^2}/m)$. Therefore, if $p<\infty$, by the Hölder inequality

$$I_1\leqslant\int_{A_m}\Delta_m^{p-1}(x)\int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)}|f'_m(t)|^p\,dt\,u_m^p(x)\,dx$$

$$\leqslant\mathscr{C}^p\int_{A_m}\int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)}|f'_m(t)|^p\,\Delta_m^{p-1}(t)\,u_m^p(t)\,dt\,dx$$

$$\leqslant\mathscr{C}^p\int_{x_1}^{x_m}\Delta_m^{p-1}(t)|f'_m(t)|^p\,u_m^p(t)\left[\int_{|x-t|\leqslant\Delta_m(x)}dx\right]dt$$

$$\leqslant\mathscr{C}^p\int_{x_1}^{x_m}\Delta_m^p(t)|f'_m(t)|^p\,u_m^p(t)\,dt,$$

i.e.,

$$I_1\leqslant\frac{\mathscr{C}^p}{m^p}\|\varphi f' u_m\|^p_{L_p([x_1, x_m])}.$$

Similarly, we have

$$I_2 = \int_{\tau_1 - m^{-1}}^{\tau_1 + m^{-1}} \left[\int_{x - (\Delta_m(x)/2)}^{x + (\Delta_m(x)/2)} |f'(t)|\, dt \right]^p u^p(x)\, dx$$

$$\leqslant \int_{\tau_1 - m^{-1}}^{\tau_1 + m^{-1}} \Delta_m^{p-1}(x) \int_{x - (\Delta_m(x)/2)}^{x + (\Delta_m(x)/2)} |f'(t)|^p\, dt\, u^p(x)\, dx$$

$$\leqslant \mathscr{C}^p \int_{\tau_1 - m^{-1}}^{\tau_1 + m^{-1}} \int_{x - (\Delta_m(x)/2)}^{x + (\Delta_m(x)/2)} \Delta_m^{p-1}(t) |f'(t)|^p v^{p\gamma, p\delta}(t)\, dt\, |x - \tau_1|^{a_1 p}\, dx$$

$$\leqslant \mathscr{C}^p \int_{\tau_1 - cm^{-1}}^{\tau_1 + cm^{-1}} \Delta_m^{p-1}(t) |f'(t)|^p v^{p\gamma, p\delta}(t) \left[\int_{\tau_1 - m^{-1}}^{\tau_1 + m^{-1}} |x - \tau_1|^{a_1 p}\, dx \right] dt.$$

Using

$$\int_{\tau_1 - m^{-1}}^{\tau_1 + m^{-1}} |x - \tau_1|^{a_1 p}\, dx \leqslant \mathscr{C} \Delta_m(t)(|t - \tau_1| + m^{-1})^{a_1 p},$$

it follows that

$$I_2 \leqslant \frac{\mathscr{C}^p}{m^p} \|\varphi f' u_m\|^p_{L_p([x_1, x_m])}.$$

Recalling the estimation for I_1, the lemma follows if $p < \infty$. The case $p = \infty$ is similar (cf. [3, p. 80]). ∎

Proof of Theorem 2.2. We recall that $S^-(f_m) \leqslant f_m \leqslant S^+(f_m)$ and $q_M^- \leqslant S^-(f_m) \leqslant q_M^+$, $P_M^- \leqslant S^+(f_m) \leqslant P_M^+$, from which $q_M^- \leqslant f_m \leqslant P_M^+$. So

$$\begin{aligned}\tilde{E}_M(f_m)_{u,p} &\leqslant \|(P_M^+ - q_M^-)\, u\|_p \\ &\leqslant \|[P_M^+ - S^+(f_m)]\, u\|_p + \|[S^+(f_m) \\ &\quad - S^-(f_m)]\, u\|_p + \|[S^-(f_m) - q_M^-]\, u\|_p \\ &\leqslant \|(P_M^+ - P_M^-)\, u\|_p + \|(q_M^+ - q_M^-)\, u\|_p \\ &\quad + \|[S^+(f_m) - S^-(f_m)]\, u\|_p.\end{aligned}$$

Using Lemma 3.2 and Lemma 3.4,

$$\tilde{E}_M(f_m)_{u,p} \leqslant \frac{\mathscr{C}}{m} \|\varphi f' u_m\|_{L_p([x_1, x_m])} + \|[S^+(f_m) - S^-(f_m)]\, u\|_p.$$

Now, being that $f\in \mathrm{AC}_{\mathrm{Loc}}$, if $x\in[y_{k-1}, y_k]$, $k=\rho, ..., \sigma$, we have

$$S^+(f_m; x)-S^-(f_m; x)\leqslant \int_{t_{k-1}}^{t_{k+1}} |f'_m(t)|\, dt \leqslant \int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)} |f'_m(t)|\, dt,$$

whence

$$\begin{aligned}\|[S^+(f_m)-S^-(f_m)]\, u\|_p^p &\leqslant \int_{-1}^{1}\left[\int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)} |f'_m(t)|\, dt\right]^p u^p(x)\, dx\\ &\leqslant \int_{y_{\rho-1}}^{x_1}\cdots+\int_{x_1}^{x_m}\cdots+\int_{x_m}^{y_\sigma}\cdots\\ &\leqslant \int_{x_1}^{x_m}\left[\int_{x-(\Delta_m(x)/2)}^{x+(\Delta_m(x)/2)} |f'_m(t)|\, dt\right]^p u^p(x)\, dx.\end{aligned}$$

Then, working as in Lemma 3.4, we have

$$\|[S^+(f_m)-S^-(f_m)]\, u\|_p \leqslant \frac{\mathscr{C}}{m}\, \|\varphi f' u_m\|_{L_p([x_1,\, x_m])},$$

whence we get

$$\tilde{E}_M(f_m)_{u,p}\leqslant \frac{2\mu a\mathscr{C}}{M}\, \|\varphi f' u_m\|_{L_p([x_1,\, x_m])}=\frac{\mathscr{C}}{m}\, \|\varphi f' u_m\|_{L_p([x_1,\, x_m])}. \quad ∎$$

Proof of Corollary 2.3. We start from

$$\tilde{E}_m(f_m)_{u,p}\leqslant \frac{\mathscr{C}}{m}\, \|f'\varphi u_m\|_{L_p([x_1,\, x_m])}.$$

Now let $Q^+(x)=\int_{-1}^{x}\pi^+(t)\, dt$ and $Q^-(x)=\int_{-1}^{x}\pi^-(t)\, dt$, where $\pi^\pm\in \Pi_{m-1}$ and $\pi^-(x)\leqslant f'_m(x)\leqslant \pi^+(x)$, $x\in[x_1, x_m]$. Therefore, we have

$$\begin{aligned}\tilde{E}_m(f_m)_{u,p} &= \tilde{E}_m(f_m-Q^-)_{u,p}\\ &\leqslant \frac{\mathscr{C}}{m}\, \|(f'-\pi^-)\, \varphi u_m\|_{L_p([x_1,\, x_m])}\\ &\leqslant \frac{\mathscr{C}}{m}\, \|(\pi^+-\pi^-)\, \varphi u_m\|_{L_p([x_1,\, x_m])}\\ &\leqslant \frac{\mathscr{C}}{m}\, \|(\pi^+-\pi^-)\, \varphi u_m\|_{L_p(B_m)},\end{aligned}$$

where $B_m = [x_1, x_m] - \bigcup_{k=1}^{r} [\tau_k - m^{-1}, \tau_k + m^{-1}]$. Here the last inequality follows from a result in [6, Lemma 2.2, p. 105]. Since

$$\|(\pi^+ - \pi^-)\varphi u_m\|_{L_p(B_m)} \leqslant \mathscr{C} \|(\pi^+ - \pi^-)\varphi u\|_{L_p(B_m)},$$

it follows that

$$\tilde{E}_m(f_m)_{u,p} \leqslant \frac{\mathscr{C}}{m} \|(\pi^+ - \pi^-)\varphi u\|_{L_p([x_1, x_m])},$$

and, making the infimum be on $\pi^{\pm}$, (2.9) follows. ▮

Proof of Theorem 2.4. We observe that, from (2.3) and Theorem 2.1,

$$\|[f - L_m(w, f)]u\|_p \leqslant \mathscr{C} \tilde{E}_{m-1}(f_m)_{u,p} + \|(f - f_m)u\|_p, \qquad 1 < p < \infty.$$

Moreover,

$$\|(f - f_m)u\|_p \leqslant \|[f - f(x_1)]u\|_{L_p([-1, x_1])} + \|[f - f(x_m)]u\|_{L_p([x_m, 1])}.$$

We estimate the first term by Minkowski inequality

$$\begin{aligned} \|[f - f(x_1)]u\|_{L_p([-1, x_1])} &= \left[\int_{-1}^{x_1} |f(x) - f(x_1)|^p u^p(x)\,dx\right]^{1/p} \\ &= \left[\int_{-1}^{x_1} \left|\int_{-1}^{x_1} (t - x)_+^0 f'(t)\,dt\right|^p u^p(x)\,dx\right]^{1/p} \\ &\leqslant \int_{-1}^{x_1} |f'(t)| \left[\int_{-1}^{x_1} (t - x)_+^0 u^p(x)\,dx\right]^{1/p} dt \\ &\leqslant \mathscr{C} \int_{-1}^{x_1} |f'(t)| \left[\int_{-1}^{t} (1 + x)^{\delta p}\,dx\right]^{1/p} dt \\ &\leqslant \mathscr{C} \int_{-1}^{x_1} |f'(t)|(1 + t)^{1/p} u(t)\,dt. \end{aligned}$$

The estimation of $\|[f - f(x_m)]u\|_{L_p([x_m, 1])}$ is similar and the theorem is proved. ▮

Proof of Corollary 2.5. The assertion follows from Theorem 2.2 and the inequalities

$$\int_{I_m} |f'(t)|(1-t^2)^{1/p}\, u(t)\, dt \leqslant \mathscr{C} \int_{I_m} |f'(t)|(1-t^2)^{1/p}\, u_m(t)\, dt$$

$$= \mathscr{C} \int_{I_m} |f'(t)|\, \varphi(t)\, u_m(t)\, \varphi(t)^{1-(2/q)}\, dt$$

$$\leqslant \mathscr{C}\, \|f'\varphi u_m\|_p \left(\int_{I_m} \varphi(t)^{q-2}\, dt\right)^{1/q}$$

$$\leqslant \frac{\mathscr{C}}{m} \|f'\varphi u_m\|_p. \quad \blacksquare$$

Proof of Corollary 2.6. We have

$$\|(f-P)\, u\|_p \leqslant \|(f-f_m)\, u\|_p + \|(f_m - P)\, u\|_p,$$

$$n \geqslant 2, \quad P \in \Pi_m, \quad p \in [1, \infty].$$

Making the infimum on P,

$$E_m(f)_{u,p} \leqslant \|(f-f_m)\, u\|_p + E_m(f_m)_{u,p}$$

$$\leqslant \|(f-f_m)\, u\|_p + \tilde{E}_m(f_m)_{u,p}.$$

From the proofs of Theorem 2.4 and Corollary 2.5 it follows that

$$\|(f-f_m)\, u\|_p \leqslant \frac{\mathscr{C}}{m} \|f'\varphi u_m\|_p, \qquad 1 \leqslant p < \infty,$$

with $\mathscr{C}$ independent on m, f, P.

Since $\|f'\varphi u_m\|_\infty < \infty$, (2.7) still holds for $p = \infty$, and Corollary 2.6 follows from Theorem 2.2. ∎

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